

SPECTRAL RADIUS INTERPOLATION AND ROBUST CONTROL

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Abstract

We have previously discussed in [12] a new kind of interpolation problem which is closely related to the multivariable gain margins of Doyle [3] and Safonov [8]. For this interpolation problem, one is required to interpolate on the disc with analytic matrices of bounded spectral radius instead of norm (as in classical H^∞ theory). In this note, we announce the mathematical solution to this problem. Full details will be found in the paper [2].

1. Introduction

A key observation is that the standard one block H^∞ optimal control problem in the finite dimensional case amounts to Nevanlinna-Pick interpolation (both for SISO and MIMO systems). Indeed, the connection between interpolation theory and certain questions involving LTI, finite dimensional plants is the simple fact that the problem of internal stabilization reduces to Lagrange-type interpolation for such plants [1], [4], [6], [7], [10], [13]. The basis of H^∞ optimization theory is that for certain design problems, one is required to interpolate on the unit disc (or equivalently, the right half plane) by analytic matrix-valued functions of bounded norm, which is precisely the Nevanlinna-Pick interpolation problem.

For SISO systems gain margin optimization [10] works in the same way in that via a conformal equivalence, one can transform the question of finding an interpolating function whose range is a certain simply-connected subset of the complex plane to one whose range is the unit disc, and hence once more derive an interpolation problem of the Nevanlinna-Pick kind. Now for MIMO systems, we have shown (see [11], [12]) that if one wants to play the same game with multivariable generalizations of gain margin ([3], [8]), one derives interpolation not with a norm constraint but with a spectral radius constraint.

A partial solution to this problem has already been given in [11] in a very special case. The purpose of this note is to announce the mathematical solution of this new kind of interpolation problem. Complete details of the results given here may be found in [2].

2. Internal Stability as Interpolation

In this section we would like to briefly show how the problem of internal stabilization for LTI, finite dimensional plants reduces to one of interpolation. See [1], [4], [6], [7], [10-12], [13] and the references

therein for more details about the facts discussed below.

Let $P(s)$ denote a $p \times m$ LTI finite dimensional plant and $C(s)$ an $m \times p$ internally stabilizing compensator. In the usual way we define the sensitivity function to be $S(s) := (I + P(s)C(s))^{-1}$. Then invoking the standard coprime factorizations we get that

$$S(s) = L - UZ$$

where L and U are completely determined by $P(s)$, U is an inner matrix-valued function, and the "free parameter" $Z \in RH_{\infty}^{m \times p}$ (= the space of $m \times p$ matrices with entries which are real rational functions bounded in the right half plane H). Now one wants to compute

$$\inf \{ \|S\|_\infty : C \text{ stabilizing compensator} \}.$$

As is well-known this problem can either be reduced to one of tangential Nevanlinna-Pick interpolation, or matricial Nevanlinna-Pick interpolation via the transformation

$$U^{alg} S = U^{alg} L - \det U I Z,$$

where U^{alg} denotes the algebraic adjoint of U , and I the identity matrix.

With these preliminary remarks made we are ready to formulate our control problem which will lead to the spectral Nevanlinna-Pick interpolation. Details of the transformations mentioned below may be found in [10], [11]. Let $P(s)$ be a $p \times m$ MIMO finite dimensional plant. Consider the following family of plants (see also [3], [8], [10-12]):

$$P_k(s) := \{kP(s) : k \in K\}$$

$$K := \{k \in \mathbb{C} : k = 1 + s, |s| \leq r\}.$$

Then using the same arguments as in [10-12], we may show that $C(s)$ internally stabilizes the closed loop for the family $P_k(s)$, $k \in K$ if and only if there exists a rational matrix-valued function $S(s)$ which is analytic and bounded in the right half plane such that $S : \hat{H} \rightarrow G$ and which moreover satisfies standard Nevanlinna-Pick interpolation conditions, where \hat{H} denotes the closed right half plane $\cup \infty$, and

$$G := \{p \times p \text{ matrices } M : \det(I + kM) \neq 0, k \in K\}.$$

But now it is easy to construct a conformal equivalence (which is a linear fractional transformation) $\phi : G \rightarrow \Omega$ where Ω denotes the space of $p \times p$ matrices with spectral radius less than one. From standard conformal mapping theory (see [10-12]) the interpolation constraints on S may be transformed to similar constraints on $\phi \circ S : \hat{H} \rightarrow \Omega$. In other words, we have a Nevanlinna-Pick type problem in which instead of bounding the norm of the interpolant we

bound its spectral radius. Notice that for $m = p = 1$ (i.e. the SISO case) we have that $\Omega = D$ (the unit disc), and hence the kind of robust stabilization problem we have been considering becomes equivalent to ordinary scalar Nevanlinna-Pick interpolation.

We should also add that a similar trick works for real parameter variations ($k \in [a, b]$ a real interval with $0 < a < 1 < b$), and we can even take parameter variation spaces of the form

$$K := \{ \text{diag}(k_1, \dots, k_p) \}$$

where the k_j 's may be either real or complex (μ in the 1×1 block case). See [11] for details about the conformal mapping theory associated to this set.

3. Solution of the Spectral Nevanlinna-Pick Problem

In this section, we very briefly outline our solution to the spectral Nevanlinna-Pick interpolation problem. For simplicity, we will consider just the matricial case. The spectral analogue of the tangential case can be handled as in [2], to which we refer the reader to all details of what follows below. Moreover, following standard mathematical practice we work on the unit disc D rather than (its conformal equivalent) the right half plane H . Accordingly, we are given n distinct points $z_1, \dots, z_n \in D$, and $n, p \times p$ matrices F_1, \dots, F_n . We want necessary and sufficient conditions for the existence of a rational matrix-valued function $F(z)$, analytic in the unit disc such that

$$F(z_i) = F_i$$

for $i = 1, \dots, n$, and such that

$$\sup \{ sr(F(z)) : z \in \bar{D} \} < 1, \quad (1)$$

where for a matrix Q , $sr(Q)$ denotes its spectral radius. We can now state the following theorem from [2]:

THEOREM 1. With the above notation $F(z)$ exists if and only if there exist invertible $p \times p$ matrices M_i for $i = 1, \dots, n$ such that

$$\left[\frac{I - M_i F_i M_i^{-1} (M_j F_j M_j^{-1})^*}{1 - \bar{z}_i z_j} \right]_{1 \leq i, j \leq n} > 0. \quad (2)$$

We would now like to discuss a bit the derivation of Theorem 1 which will also give us a numerical scheme for the theorem's implementation. The proofs of these results can all be found in [2]. Let H denote a separable Hilbert space. Then a contraction $A : H \rightarrow H$ is a linear bounded operator such that $\|A\| \leq 1$. Now given two contractions T and A on H , we set

$$\rho_T(A) := \inf \{ \|MAM^{-1}\| : M \text{ is invertible and } MT = TM \}.$$

The quantity $\rho_T(A)$ is called the T -spectral radius of A . One can show that $\rho_T(A)$ enjoys the following properties [2]:

- (i) $\rho_A(A) = sr(A)$ for all $a \in \mathbb{C}$, where I denotes the identity operator on H .
- (ii) $sr(A) \leq \rho_T(A) \leq \|A\|$.
- (iii) If $T = S$ is a unilateral shift in H , then $\rho_S(A) = sr(A)$ for every $A : H \rightarrow H$ such that $AS = SA$.

Now given $m \in H^\infty$ an all-pass (inner) rational function, set $H(m) := H^2 \ominus mH^2$, and $H := H(m) \oplus \dots \oplus H(m)$ (direct sum of p copies of $H(m)$). (All of our Hardy spaces will be defined on the unit disc D in the standard way.) For S the unilateral shift on H^2 , and for $P_{H(m)} : H^2 \rightarrow H(m)$ orthogonal projection, let $S(m) := P_{H(m)} S|_{H(m)}$ denote the compressed shift. We set $T := S(m) \oplus \dots \oplus S(m)$, and $U := S \oplus \dots \oplus S$ on $K := H^2 \oplus \dots \oplus H^2$ (direct sums of p copies). We can state the following spectral version [2] of the commutant lifting theorem [9]:

THEOREM 2. Taking T, U, H, K as just defined, let $A : H \rightarrow H$ be any contraction such that $AT = TA$. Then we have that

$$\rho_T(A) :=$$

$$\inf \{ sr(B) : B \text{ is a commuting dilation of } A, \text{ i.e., } BU = UB, P_{H\mathcal{B}}|_H = AP_{H\mathcal{B}} \}$$

where $P_{H\mathcal{B}} : K \rightarrow H$ denotes orthogonal projection.

REMARKS.

First of all, note that Theorem 1 may be derived as a corollary of Theorem 2 by taking

$$m := \prod_{i=1}^n \frac{z - z_i}{1 - \bar{z}_i z}.$$

(See [2] for the details.) Next in [2], it is shown that we can always minimize over rational B . Moreover, under the hypotheses of Theorem 2 (in which we are dealing with finite block diagonal matrices), one could use the gradient search procedures described in [3] and [5] to compute "scalings" M arbitrary close to the optimal. Then if one invertibly dilates such a (nearly) optimal M to an analytic matrix $M(z)$, one is required to check the validity of matrix inequality (2) for $M_i := M(z_i)$ ($i = 1, \dots, n$), i.e. via these scalings we are reduced to a classical matricial Nevanlinna-Pick problem. (Equivalently, an interpolating $F(z)$ exists satisfying (1) if and only if for such an M , we have that $\|MAM^{-1}\| < 1$, for A as in Theorem 2.)

In short, via the chain of arguments given above and in [3], [5] we have an implementable procedure for checking the optimal gain margin. Mathematical details of the spectral commutant lifting theorem will appear in [2].

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